Lecture 2: Production Economy w/o aggregate uncertainty

The Aiyagari model

Jochen Mankart

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Production economy with exogenous labor productivity risk (Aiyagari 1994)

1. Similar set-up as in Huggett (1993)
2. Now production is endogenous.
   - Standard neoclassical production function (Cobb-Douglas)
   - Firms demand capital and labor.
3. Questions addressed
   - How does consumption saving decision depend on: wealth, income state, uncertainty about income.
   - What are equilibrium capital stock and interest rate?
4. We focus on 2 new things:
   1. How to get firms' into the model, easy.
   2. How to deal with continuous labor productivity process, discretize it.

Ingredients (I)

Households

- Unit mass of infinitely lived agents
- Utility function

$$E_0\left[\sum_{t=0}^{\infty}\beta^t u\left(c_t\right)\right] \tag{1}$$

- individual labor productivity (no aggregate uncertainty): n states Markov process

$$s_t \in S = \left\{s_t^j\right\}_{j=1}^n$$

with transition

$$\pi\left(s'|s\right) = prob(s_{t+1} = s'|s_t = s)$$

and associated earnings

$$y\left(s^j\right) = w * s^j$$

Ingredients (II)

Financial market

Financial markets

- One period discount bonds with price

$$q_t = \frac{1}{1 + r_t}$$

- since no aggregate uncertainty, equilibrium price constant

$$q_t = q = \frac{1}{1 + r}$$

- Borrowing constraint (potentially NBL)

$$\underline{a} \leq 0$$

so feasible budget set

$$A = \{a_t \in \mathbb{R} : a_t \geq \underline{a}\}$$

Ingredients (III)

Firms

- Aggregate production technology Cobb-Douglas

$$Y_t = K_t^{\alpha} L_t^{(1-\alpha)}$$

where L_t represents efficiency units. (TFP normalized to 1.)

- In optimum, marginal conditions are

$$r_t = \frac{\partial Y_t(\cdot)}{\partial K_t} - \delta \quad (2)$$

$$w_t = \frac{\partial Y_t(\cdot)}{\partial L_t} \quad (3)$$

where δ is depreciation rate and firms take prices and wages as given.

- This yields a capital demand function $k^d = k(w_t, r_t)$

Solution

- Household problem can again be solved by value function.
- Firms is static in each period.
- What is market clearing condition now?

Equilibrium

Definition

A recursive stationary equilibrium is an allocation (c, a'), a (constant) interest rate r^*, a constant wage w^* and an invariant distribution ψ^* for which

1. given r^*, households optimize, i.e. (c, a') are optimal decision rules (solve (**??**))
2. given r^* and w^* firms optimize, i.e. (2) , (3) hold.
3. capital used in production$K$ is provided for by households
4. $\int_{A,S} a'(a, s; q)\, d\psi^* = K$
5. ψ^* is the stationary probability measure $\psi^* = F\psi^*$.

Storesletten, Telmer and Yaron (2004, JME)

1. They show that both, income and consumption inequality grow over the life-cycle
2. Main sources of inequality:
   1. Fixed effect determined before agent enters labor force (age 16)
   2. Persistent shocks to earnings during life.
3. Complete markets model unlikely to be able to explain this.

Earnings estimation

- Idiosyncratic component of log-earnings of household i of age h:

$$\begin{aligned} u_{i,h} &= \alpha_i + \varepsilon_{i,h} + z_{i,h} \qquad (4) \\ z_{i,h} &= \rho z_{i,h-1} + \eta_{i,h} \qquad (5) \end{aligned}$$

where

$\alpha_i \sim N\left(0, \sigma_\alpha^2\right), \varepsilon_i \sim N\left(0, \sigma_\varepsilon^2\right), \eta_{i,hi} \sim N\left(0, \sigma_\eta^2\right), z_{i,0} = 0$

  - α_i fixed effect determined at birth (or better at beginning of economics life)
  - $z_{i,h}$ persistent life cycle shock
  - $\varepsilon_{i,h}$ transitory life cycle shock
  - ρ AR(1) coefficient, measures persistence

- Key finding:

$$\widehat{\rho} > 0.95$$

Shocks are very persistent, maybe even permanent.

Approximating persistent labor productivity, Adda Cooper pp.56

- In value function, we take conditional expectation with respect to tomorrow's shock realizations

$$E_t\left(z_{t+1}\right)=\int z_{t+1}dF\left(z_{t+1}\middle|z_t\right)$$

- Idea of all numerical integration techniques is to replace infinite sum (integral) with some finite sum.
- We need 2 things:
  1. Nodes (points where we calculate the integral)
  2. Weights (how important is that point)

$$\int z_{t+1}dF\left(z_{t+1}\middle|z_t\right)\simeq w_1z_1+w_2z_2+\ldots w_Nz_N$$

where N is number of nodes.

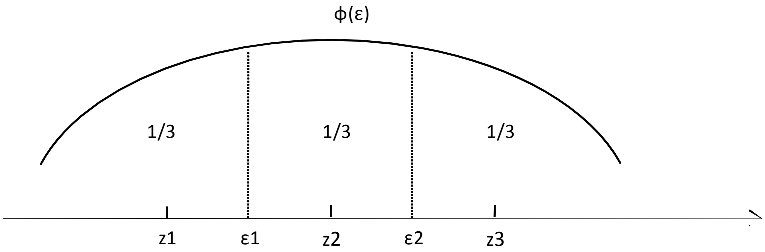
Adda Cooper pp.56 (overview)

- ε_t follows AR(1) with mean μ and autocorrelation ρ

$$\begin{aligned} \varepsilon_t &= (1-\rho)\,\mu + \rho\varepsilon_{t-1} + u_t \\ u_t &\sim N\left(0, \sigma^2\right) \end{aligned}$$

- Steps
    1. Discretize process into N intervals
    2. For each interval, get conditional mean
    3. Get conditional transition probability of moving from one interval to the next

Adda Cooper pp.56 (illustration) for 3 nodes



Step 1: construct intervals

- N intervals, this $N+1$ $\varepsilon's$, with $\varepsilon^0 = -\infty$ and $\varepsilon^N = \infty$
- Construct such that all have equal probability.
- By normality of u_i, the cut off points $\left\{\varepsilon^i\right\}_{i=0}^N$ are given by

$$\Phi\left(\frac{\varepsilon^{i+1}-\mu}{\sigma_\varepsilon -}\right) - \Phi\left(\frac{\varepsilon^{i}-\mu}{\sigma_\varepsilon}\right) = \frac{1}{N}, \qquad i = 0, \ldots, N-1$$

where Φ is cdf and $\sigma_\varepsilon = \frac{\sigma}{\sqrt{1-\rho^2}}$

$$\varepsilon^i = \sigma_\varepsilon \Phi^{-1}\left(\frac{i-1}{N}\right) + \mu$$

Step 2: conditional mean

$$
\begin{aligned}
z^i &= E\left(\varepsilon_t \middle| \varepsilon_t \in \left[e^i, \varepsilon^{i+1}\right]\right) \\
&= \sigma_e \frac{\phi\left(\frac{\varepsilon^i - \mu}{\sigma_\varepsilon}\right) - \phi\left(\frac{\varepsilon^{i+1} - \mu}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\varepsilon^{i+1} - \mu}{\sigma_\varepsilon -}\right) - \Phi\left(\frac{\varepsilon^i - \mu}{\sigma_\varepsilon}\right)} + \mu \\
&= N\sigma_e \left[\phi\left(\frac{\varepsilon^i - \mu}{\sigma_\varepsilon}\right) - \phi\left(\frac{\varepsilon^{i+1} - \mu}{\sigma_\varepsilon}\right)\right] + \mu
\end{aligned}
$$

Transition probabilities

$$\pi_{i,j} = P\left(\varepsilon_t \in \left[e^j, \varepsilon^{j+1}\right] \left| \varepsilon_{t-1} \in \left[e^i, \varepsilon^{i+1}\right]\right.\right)$$

$$\pi_{i,j} = \frac{N}{\sqrt{2\pi\sigma_\varepsilon^2}} \int_{\varepsilon^i}^{\varepsilon^{i+1}} e^{\left(-\frac{(u-\mu)^2}{2\sigma_\varepsilon^2}\right)} \left[\begin{array}{c} \Phi\left(\frac{\varepsilon^{j+1}-\mu(1-\rho)-\rho u}{\sigma}\right) - \\ \Phi\left(\frac{\varepsilon^{j}-\mu(1-\rho)-\rho u}{\sigma}\right) \end{array} \right] du$$

the integral wrt u can be solved numerically.

Use

- Test: What is $\pi_{i,j}$ if $\rho = 0$?
- Suppose we want to replicate Storesletten et al's paper, do we have to write the code for approximation of AR(1) ourselves?
- No, just google it, e.g. http://www.eco.utexas.edu/~cooper/dynprog/dynprog-programs.html
- or http://www2.hhs.se/personal/floden/
- or for very persistent process $\rho > 0.95$ www.karenkopecky.net/

Goal: How can we adapt our Huggett code to make it solve Aiyagari model?

- 2 new features:
  1. labor income process
  2. aggregate production function
- (2) is easy:
  - Keep guessing interest rate
  - due to Cobb Douglas, we get capital -labor ratio and so wage.
- (1) is easy too
  - Decide on number of nodes with which to approximate AR(1)
  - Call some discretization routine to get nodes and transition probabilities.
  - Run.

PRACTICE SESSION 2

Interest rate and saving rate (table II)

Coefficient of variation 0.2

$\rho \backslash crra$	1	3	5
0	4.16/23.67	4.14/23.71	4.09/23.83
0.3	4.14/23.73	4.04/23.91	3.91/24.19
0.6	4.09/23.82	3.88/24.25	3.59/24.86
0.9	3.93/24.14	3.29/25.51	2.53/27.36

- 1.number r (in %) / 2.number saving rate (in %)
- Complete markets counterparts 4.17 and 23.67
- Idiosyncratic productivity risk has modest effects. But these increase with persistence and risk aversion.
- Income and wealth inequality way too low.