Lecture 3: Production Economy w/ aggregate uncertainty

Krusell Smith model

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Motivation

- Standard business cycle models (RBC) use representative agent assumption. This would be ok:
  - if we observed complete financial markets (Arrow Debreu or Arrow securities);
  - if these models behaved in the same way as more realistic heterogenous agent models (which are much harder to solve).
- To talk about cyclical properties of income and/or wealth distribution, one (obviously) needs a heterogenous agent model.

Overview

- Agents face idiosyncratic shocks: employed/unemployed (Huggett,Imrohogolu)
- General equilibrium model with aggregate uncertainty: w and r fluctuate (new)
- No insurance markets, no borrowing
- Agents can only self -insure through holding capital (as in Aiyagari)

Consumers (I)

- Utility function

$$E_0\left[\sum_{t=0}^{\infty}\beta^t u\left(c_t\right)\right] \quad (1)$$

$$u\left(c\right) = \frac{c^{1-\sigma}-1}{1-\sigma} \quad (2)$$

- Employment state ε: either e (employed) or u (unemployed)
- $$y_t = \begin{cases} y & \text{if} \quad \varepsilon = e \\ 0 & \text{if} \quad \varepsilon = u, \end{cases}$$
- Aggregate states

$$z_t = \begin{cases} z_g & \text{good state} \\ z_b & \text{bad state} \end{cases}$$

- with Markov transition matrix Π_z

Consumers (II)

- Unemployment rate u_t depends on aggregate state z_t
- Aggregate states

$$u_t = \begin{cases} u_g & \text{good state} \\ u_b & \text{bad state} \end{cases}$$

therefore individual transition matrix is time varying with Markov transition matrix Π

- Denote probability to move from (z_s, ε) to $(z_{s'}, \varepsilon')$ as $\pi_{ss',\varepsilon\varepsilon'}$

Production

- Technology

$$y_t = z_t \bar{k}_t^{\alpha} \bar{l}_t^{1-\alpha}$$

- perfect competition pins down w_t and r_t
- Note that there is no uncertainty concerning l_t
- What are state variables for individual agents problem?

Aggregate state

- Aggregate state is

$$(\Gamma, z)$$

  where Γ is distribution of agents over capital holdings and employment status.
- between periods Γ will change, so we also need a law of motion for it.

$$\Gamma' = H\left(\Gamma, z, z'\right)$$

- Why do we need this?
- To predict future (factor) prices! Think of EE.

Value function

- How many state variables?
- Value function

$$v\left(k,\varepsilon;\Gamma,z\right)=\max\left\{u\left(c_t\right)+\beta\mathbb{E}\left[v\left(k',\varepsilon';\Gamma',z'\right)\left|\varepsilon,z\right]\right\}\quad(3)$$

- subject to

$$\begin{aligned}c+k' &= r\left(\cdot\right)k+w\left(\cdot\right)\varepsilon+\left(1-\delta\right)k\\ \Gamma' &= H\left(\Gamma,z,z'\right)\\ k' &\geq 0\end{aligned}$$

- Denote savings function

$$k'=f\left(k,\varepsilon;\Gamma,z\right)$$

Recursive competitive equilibrium

Definition

A RCE is a law of motion H, a pair of individual functions v and f, and pricing functions (r, w) such that:

- (v, f) solve (3)
- interest rate r and wage w are competitive (solve FOC of firm)
- H is generated by f, thus summing individual savings decision gives aggregate savings.

Main issue

- What do we do with Γ and $H(\cdot)$?
- These are high-dimensional objects! We would need just as many state variables. Curse of dimensionality!
- Krusell-Smith solution: Bounded rationality. Agents use only m moments to describe the distribution.
- In particular, KS show that $m = 1$ yields already very accurate results. That has been coined "approximate aggregation".

Forecasting rule for m=1

Specifically, agents use the following rule to forecast future capital

$$\begin{aligned} z &= z_g : \log \bar{k}' = a_0 + a_1 \log \bar{k} \\ z &= z_b : \log \bar{k}' = b_0 + b_1 \log \bar{k} \end{aligned}$$

i.e. $H(\cdot)$ is log-linear.

Value function used

- Agents solve following problem

$$v\left(k,\varepsilon;\bar{k},z\right)=\max\left\{u\left(c_t\right)+\beta\mathbb{E}\left[v\left(k',\varepsilon';\bar{k}',z'\right)|\varepsilon,z\right]\right\} \quad (4)$$

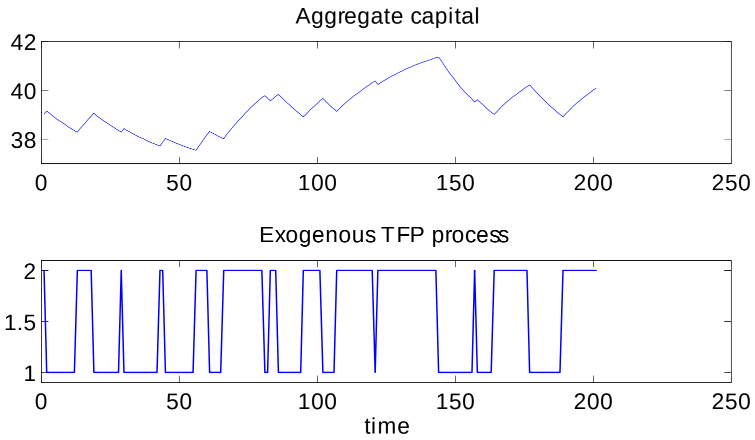
- subject to

$$\begin{aligned}
c+k' &= r\left(\cdot\right)k+w\left(\cdot\right)\varepsilon+\left(1-\delta\right)k \\
\Gamma' &= H\left(\Gamma,z,z'\right) \\
k' &\geq 0 \\
\log\bar{k}' &= a_0+a_1\log\bar{k} \quad \text{if } z=z_g \\
\log\bar{k}' &= b_0+b_1\log\bar{k}\ldots\text{if } z=z_b
\end{aligned}$$

Main challenge

- What is relation between individual savings k and aggregate savings $\bar{k}$?
- Main computational challenge is this loop:
  1. To obtain individual k', we need law of motion for aggregate $\bar{k}$
  2. But $\bar{k}$ must be result of aggregating individual $k's$ thus we have to take into account Γ.
  3. Thus we have to iterate on the forecasting rules until for given rule, and given individual decisions, the aggregate $\bar{k}$ is consistent with the individual k' (and of course the distribution of agents Γ).

Aggregate capital over time



Why does approximate aggregation hold? Why does only mean matter

KS paper, p.877 result
Good times

$$\log \bar{k}' = 0.095 + 0.962 \log \bar{k}; \quad R^2 = 0.999998, \widehat{\sigma} = 0.0028\%$$

Bad times

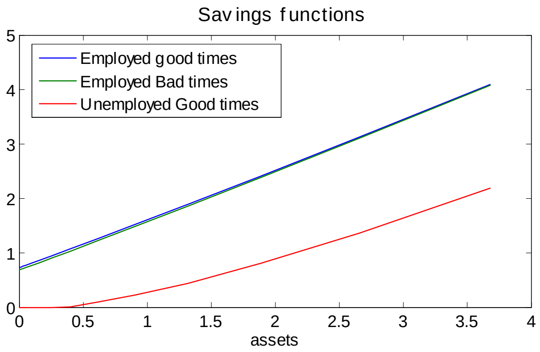
$$\log \bar{k}' = 0.085 + 0.965 \log \bar{k}; \quad R^2 = 0.999998, \widehat{\sigma} = 0.0036\%$$

where $\widehat{\sigma}$ is standard deviation of regression error.

Accuracy

- DenHaan (2010) shows that neither R^2 nor $\widehat{\sigma}$ are good measures of accuracy. He proposes several alternatives.
- The crucial point is that one should not use the aggregate law of motion when checking accuracy but only aggregate the individual policy functions.
- KS did something similar.
- If you write a paper, don't report just the R^2, check DenHaan's paper.

Figure 2: Individual decision rules in good state, for given aggregate capital.



- Aggregate state hardly matters. Savings differ only significantly between employed and unemployed!

Only mean matters

- When would aggregation be perfect?
- If propensity to save out of wealth would be identical. Then a redistribution of wealth would have no impact on savings, only mean would matter.
- In previous figure, we saw that marginal propensities to consume are almost identical for most agents, except very poor.
- But do poor matter for aggregate wealth? No, by definition of being poor, their wealth holding is negligible.
- Therefore, we get approximate aggregation: All macro variables can be described by: mean of wealth distribution and aggregate TFP.

Further comments

- Wealth distribution is way too even
- Section IV, they match wealth distribution by heterogeneity in β

$$\beta_1 = 0.9858 \quad \beta_2 = 0.9894, \quad \beta_3 = 0.9930$$

- gets wealth distribution right, all wealth held by rich, so forecasting rules work again with mean only.
- But aggregate consumption now depends a lot on poor who behave as "hand-to-mouth" consumers. Thus PIH does not hold in this model in contrast to representative agent model.

KS code

Special issue of JEDC 34 (2010) on solution methods for KS.

1. It compares 6 different algorithms.
2. All codes are online: 4 in matlab 2 in fortran
3. We look at Maliar, Maliar & Valli (2010) since it is closest to original KS and suitable for us
4. DenHaan & Rendahl seems better (accuracy, speed) but it involves some numerical concepts we haven't discussed. But, if you write a paper...

New numerical tool: simulation

- In K-S, there are 2 sources of uncertainty
  1. Aggregate technology can be good or bad
  2. Individual can be employed or unemployed
- In other papers, idiosyncratic labor productivity can take on different values.
- Sometimes, we have to simulate histories of agents to compute some variables of interest

How?

- Suppose we have Markov process with grid points $\{z^i\}$ and transition matrix $\pi_{i,j} = \Pr ob(y_t = z^j | y_{t-1} = z^i)$
- Every program has a (pseudo) random number generator. From this we draw uniform u in $[0, 1]$
- Define simulation length, e.g. $T = 1000$
- Initialize process somewhere, $z_1 = z^i$
- Now, we want $z_2, z_3, ...z_T$

Solution method

- So far, we have solved our problems by finding the value function, i.e. value function iteration or endogenous gridpoints.
- Maliar et al (2010) use FOC but not EGM.
- If Euler equation holds everywhere we have a solution.

Maliar et al 2010

Key equations

- Consumption Euler eqn

$$c^{-\gamma} - h = \beta E_t \left[\left(c'\right)^{-\gamma} \left(1 - \delta - r'\right) \right] \qquad (5)$$

where h is Lagrange multiplier stemming from no borrowing constraint.

- asset (capital holding) evolution (aka budget constraint)

$$k' = (1 - \tau) w\varepsilon + \mu w (1 - \varepsilon) + (1 - \delta + r) k - c \qquad (6)$$

where indicator $\varepsilon = 1$ employed, $\varepsilon = 0$ unemployed

- complementary slackness on borrowing constraint

$$hk' = 0 \qquad h \geq 0 \qquad k' \geq 0$$

Solution based on policy function (I)

- Solve (5) for c

$$c = \left\{ h + \beta E \left[\frac{(1 - \delta - r')}{(c')^{\gamma}} \right] \right\}^{-1/\gamma}$$

- Use BC (6) solve for c

$$c = (1 - \tau) w\varepsilon + \mu w (1 - \varepsilon) + (1 - \delta + r) k - k'$$

- forward 1 period

$$c' = (1 - \tau') w'\varepsilon' + \mu' w' (1 - \varepsilon') + (1 - \delta + r') k' - k''$$

- both into rewritten EE.

Solution based on policy function (II)

yields eqn (5) in paper

$$k' = (1-\tau)\, w\varepsilon + \mu w\,(1-\varepsilon) + (1-\delta+r)\, k -$$
$$\left\{ h + \beta E\left[\frac{(1-\delta-r')}{\left((1-\tau')\, w'\varepsilon' + \mu' w'\,(1-\varepsilon') + (1-\delta+r')\, k' - k''\right)^{\gamma}} \right] \right\}^{\frac{-1}{\gamma}}$$

- Note that we have k' and k''
- This is a non linear functional equation which Maliar et al solve iteratively.

Further issues

- Factor prices and tax rate are time varying.
- Since TFP takes only 2 values, E and U take only 2 values as well.
- To forecast future factor prices, they use only 1st. moment of capital distribution.
- Policy function is a 4 dimensional array
- individual capital holdings, continuous in theory, approximate with 100 grid points, k
  1. aggregate capital stock, continuous in theory, approximate with 4 points m
  2. aggregate employment, 2 states only so 2 nodes. ε
  3. TFP also only 2 nodes. a

Maliar et al's algorithm (I)

Overview

1. Guess aggregate law of motion

$$\ln k' = A_i + B_i \ln k \qquad i = G, B \tag{7}$$

implies future factor prices.

2. Solve individual agents' consumption-savings problem, get savings functions.
3. Use exogenous shocks to
   1. aggregate TFP and
   2. individual employment/unemployment
4. to simulate N agents over T periods, $N = 10000$, $T = 1100$

Maliar et al's algorithm (II)

Overview

5. In each period sum the individual asset holdings to get time series for aggregate capital k
6. Split this time series into G and B, run 2 regressions (eqn 7) to get new ALM

$$\ln k' = A_i^{update} + B_i^{update} \ln k \qquad i = G, B \tag{8}$$

- If regression coef. stop changing, finished, else go back to 2.

Maliar et al's algorithm - Detail

Individual agents' problem

1. Guess the initial savings function $k'(k, m, \varepsilon, a)$
2. at each node, sub in guessed $k'(k, m, \varepsilon, a)$ on RHS of eqn (5), assume associated LM $h(k, m, \varepsilon, a) = 0$. This yields new savings function.
3. At nodes where constraint binds LM $h(k, m, \varepsilon, a) > 0$, set savings equal to borrowing constraint, i.e. this agent will eat all he can.
4. Update savings function, go back to 1, until converged.